STRUCTURED UNCERTA INTY IDENTIFICATION AND CONTROL SYNTHESIS

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Abstract

This paper presents a systematic method to design robust controllers from experimental data so as to ensure that the controller is robust with respect to all plants which cannot be discounted based on the data (to within a specified statistical confidence). The present paper extends previous multi variable results to the structured uncertainty case and demonstrates the benefit to control synthesis of estimating uncertainty bounds in structured form.

1 INTRODUCTION

In recent years, control analysis and synthesis methods have been developed which ensure controller performance and robustness with respect to norm-hounded perturbations in the plant description. For linear time-invariant (LTI) plants, such perturbations represent the designers lack of knowledge about the true plant model. For linear time-varying (I, TV) plants such perturbations must also capture temporal variation in the plant dynamics.

In the case of LTI plants, uncertainties can be substantially reduced using experimental data and system identification methods. This has led researchers to develop identification methods which directly support robust control design. Such methods involve the estimation of a "plant set" rather than a point estimate, and must produce uncertainty bounds in a form which can be incorporated into robust control formulations.

Various methods of plant set estimation have been given in the literature. The various approaches can be roughly divided depending upon whether they use time-domain estimation as in Kosut [20], Younce and Rhors [26], Goodwin and Salgado [18], or frequency domain estimation as in Lamaire et. al. [21], Parker and Bitmead [22], Bayard [4], or De Vries and Van den Hof [11][12]. Methods within each category generally differ based on the types of inputs allowed (e.g., second-order stationary, white noise, periodic, persistent (excitation, etc.), the types of quantities being estimated (e.g., plant dynamics, noise PSD's, unmodelled dynamics), model parametrization (e.g., pole-zero models, FIR models, etc.), assumptions on the noise (e.g., bounded noise, bounded noise DFT, Gaussian noise, etc.), and the type of a-priori information required (i.e., smoothness priors, open-loop damping, model order, relative degree, etc.).

Related approaches which give hard bounds on the identified model error can also be found in Helmicki, Jacobson and Nett [19] and Gu and Khargonekar [16]. However, these methods are not directly comparable since they start by assuming that frequency data is available in a specific form (i.e., with hard error bounds).

The plant set estimation method used in this paper has its roots in the approaches of Bayard [4] and de Vries and Van den Hof [12], which utilize periodic and multisinusoidal input excitation in combination with frequency domain identification techniques. The particular plant set estimation method used in this paper was developed in Bayard [5], and represents an extension of the multivariable results in [7] to the "structured uncertainty" case. Structuring may be useful for reducing conservatism in practical multivariable problems, particularly if there are variations in the signal-to-noise properties between different channels. Such variations arise from differences in plant dynamics, excitation methods, data length constraints, actuator power constraints, noise coloring characteristics, etc., and should be expected in most MIMO applications.

Using multivariable structured plant set estimation, this paper advances a method to design robust controllers from experimental data so as to ensure that the controller is robust with respect to all plants which cannot be discounted based on the data (to within a specified statistical confidence). A special case of this unified estimation and control approach has been demonstrated earlier for single-input/single systems in [6], and for multivariable systems with unstructured uncertainty in [9]. The present paper extends these results to the multivariable *structured* uncertainty case, and demonstrates the benefits to control synthesis of estimating uncertainty bounds in structured form.

Section 2 provides a formulation of the problem and outlines basic analytical assumptions. Section 3 and Section 4 respectively present analysis for nonparametric and parametric characterizations of the structured uncertainty. Section 5 provides an incorporation of the uncertainty characterization into robust control design as illustrated by the two-car problem analyzed in Section 6. Final conclusions are given in Section 7.

2 PROBLEM FORMULATION

2.1 BACKGROUND

Consider the discrete-time multivariable system with output noise, given by,

$$y(\tau) = \mathcal{P}(z^{-1})u(\tau) + v(\tau) \tag{1}$$

where $\mathcal{P}(z^{-1})$ is the n_u-input, rig-output multivariable LTI plant, $v \in \mathcal{R}^{n_y}$ is an output vector disturbance, and $\tau = 1, 2, ..,$ denotes the discrete time index. It is desired to identify this system in the following form,

$$\mathcal{P}(z^{-1}) = P^o(z^{-1}) + \Delta_A(z^{-1}) \tag{2}$$

where $P^o(z^{-1})$ is a nominal estimate of the true plant $\mathcal{P}(z^{-1})$, and $\Delta_A(z^{-1})$ is the additive uncertainty defined as $AA = \mathcal{P} - P^o$. Since the true plant is not known, it is desired to represent the additive uncertainty in the form

$$AA = W_{A2}\Delta W_{A1} \tag{3}$$

such that A is a structured norm-bounded perturbation (to be described in more detail later), and W_{A1} and W_{A2} are stable rational transfer function matrices. The weighings W_{A1} and W_{A2}

are then typically incorporated into the control design, to ensure robustness properties over the additive uncertainty set.

This paper presents a method for identifying a nominal plant estimate P^o, and weighting filters W_{A1}, W_{A2}, from experimental data, such that the relation $P = P^o + W_{A2}\Delta W_{A1}$ holds (for some structured $||\Delta||_\infty \leq 1$) to a specified statistical confidence $1 - \alpha$ specified by the designer. These quantities are then used to directly synthesize a robust controller C using standard packages such as Balas et. *al.* [2], and Chiang and Safonov [1 O]) such that the closed-loop system has desirable stability and performance properties for all plants in the u ncertainty set,

The rationale is that if C can ensure some level of performance for all plants in this additive uncertainty set, then the controller will work *as designed when implemented on the real plant* with probability $1 - \alpha$. This approach effects a marriage between the hard uncertainty bounds used in modern H_∞ robust control designs, and the soft bounds obtainable using statistical methods.

2.2 ASSUMPTIONS ON A-PRIORI INFORMATION

The estimation of a plant set requires the specification of certain a-priori information. The assumptions are given explicitly in this section.

Let" $\bar{\sigma}(X)$ denote the maximum singular value of a matrix X. The following definition will be needed.

DEFINITION 1 A MIMO linear time-invariant (LTI) system with transfer function $G(z^{-1})$ is said to be in D(M, ρ) if the impulse response matrix sequence $\{g(\tau T')\}_{\tau=0}^{\infty}$ defined by the Z-transform relation $\sum_{\tau=0}^{\infty} g(\tau T')z^{-k} = G(z^{-1})$ satisfies,

$$\bar{\sigma}(g(\tau T')) \leq M\rho^{\tau} \tag{4}$$

for some $\infty > M > 0$ and $1 > \rho \geq 0$. ■

Simply stated, D(M, ρ) is comprised of all LTI systems whose impulse response decays exponentially. The main usefulness of Definition 1 is due to the next lemma.

LEMMA 1 Let $G(z^{-1}) \in$ D(M, p). Then the derivative of G on the unit circle can be uniformly bounded from above as follows,

$$\bar{\sigma}\left(\frac{dG(e^{-j\omega T'})}{d\omega}\right) \leq \frac{T'M\rho}{(1-\rho)^2} \tag{5}$$

PROOF: see Bayard *et. al.* [8]. ■

The bound in Lemma 1 insures a certain *smoothness* in G and allows one to overbound errors incurred interpolating *in-between* grid points.

ASSUMPTION 1 The true plant $\mathcal{P}(z^{-1})$ is a stable unknown linear time-invariant (1,1'1) n_u-input n_y-output multivariable transfer function assumed to be in $\mathcal{D}(M,\rho)$, where M and p are assumed known. •

The experiment design is now briefly described. Consider the periodic. input excitation design into the nth input, composed of a harmonically related sum of sinusoids,

$$u_s(\tau, n_1) = \sum_{\tau=1}^{n_s} \sqrt{2\alpha_k(n_1)} \cos(\omega_k \tau T + \phi_k(n_1)) \tag{6}$$

where T is the sampling period, $\omega_k = 2\pi k/T_p$, $T_p = N_s T$, $n_s \leq N_s/2 - 1$. For efficient computation using a Fast Fourier Transform (FFT) the total number of frequency grid points N_s should be chosen as a power of 2.

The power is assumed to be normalized as,

$$\sum_{k=1}^{n_s} \alpha_k(n_1) = 1 \tag{7}$$

where the relative power in each component $\{\alpha_k(n_1) > 0, k = 1, \ldots n_s\}$ is assumed specified. In order to minimize peaking in time domain the sinusoids are phased according to Schroeder [25] as,

$$\phi_k(n_1) = 2\pi \sum_{j=1}^{k} j\alpha_j(n_1) \tag{8}$$

(Here, a slightly modified form of the Schroeder phase is used in (8), as derived in Young and Patton [27]). More recent expressions which use the Schroeder design as a starting point for further reducing the crest factor of the multisinusoid signal (6) can also be used (Guillaume et. *al.* [17]). The Schroeder phasing (8) is useful for implementations which must make the most efficient use of input power subject to actuator saturation constraints. However, the actual choice of phase does not effect the analysis or change any of the main results herein.

ASSUMPTION 2 Data for the multivariable case is assumed to be taken by performing n_u separate single-input multiple-output (SIMO) experiments, using a multisinusoidal excitation of the form (6) with the full number of sinusoids $n_s = N_s/2 - 1$ for each experiment. ■

ASSUMPTION 3 The output disturbance $v(\tau) \in \mathcal{R}^{n_y}$ can be represented by $v(\tau) = W(z^{-1})d(\tau)$ where $d(\tau) \in \mathcal{R}^{n_y}$ is a white zero-mean Gaussian vector noise sequence normalized such that $E[d(i)d^T(j)] = \delta_{ij} \cdot I$; $W(z^{-1})$ is a *diagonal matrix* of filters

$$W(z^{-1}) = Diag\left\{W(z^{-1}, 1), \ldots, W(z^{-1}, n_y)\right\} \tag{9}$$

where $W(z^{-1}, n_2)$ is a minimum phase (stable and stably invertible) transfer function, $n_2 = 1, \ldots, n_y$. ■

ASSUMPTION 4 Data from each SIMO experiment is taken while the system is in periodic steady-state. ■

ASSUMPTION 5 The input period $N_s T$ of the multisinusoidal design (6) is long compared to the time constants of noise filter W and its inverse W^{-1}. ■

In this formulation, the designer has the freedom to choose the frequency shaping $\{\alpha_k(n_1)$ and the number of periods of data collected $m(n_1)$ in each SIMO experiment.

3 NONPARAMETRIC STRUCTURED UN CERTAINTY

3.1 STRUCTURED UNCERTAINTY DESCRIPTION

An additive error $\Delta_A(z^{-1})$ is used to characterize the mismatch between the true plant $\mathcal{P}(z^{-1})$ and a nominal plant estimate $P^o(z^{-1})$, i.e.,

$$\Delta_A(z^{-1}) = \mathcal{P}(z^{-1}) - P^o(z^{-1}) \tag{lo}$$

It is desired to consider the structure of Δ_A in more detail. The following definition will be useful.

DEFINITION 2 A partition of a matrix G <1<"""""" is defined as any set of integers,

$$\mathcal{X} = \{p, q; \eta_1, \ldots, \eta_p; \nu_1, \ldots, \nu_q\} \tag{11}$$

such that $\sum_{i=1}^{p} \eta_i = n_y$ and $\sum_{i=1}^{q}$ vi = n.. From the set of integers in $\mathcal{X}$, and the matrix G, one can uniquely define the *set of partitioned blocks,*

$$\mathcal{K}(\mathcal{X}, G) = \{[G]_{ij} \in \mathcal{R}^{\eta_i \times \nu_j} \; i = 1, \ldots, p, \; j = 1, \ldots, q\} \tag{12}$$

by using the construction shown in Figure 1. Specifically, the matrix G is partitioned vertically and horizontally into *p* and g segments, respectively, where the ith vertical segment is η_i rows tall, and the jth horizontal segment is ν_j columns wide. Finally, an *index set* $\kappa_{ij}(\mathcal{X})$ is defined which contains indices of all elements in the ijth block of a partition $\mathcal{X}$, i.e.,

$$\kappa_{ij}(\mathcal{X}) = \{(n_2, n_1) : G_{n_2 n_1} \text{ is an element of block } [G]_{ij} \in \mathcal{K}(\mathcal{X}, \text{G})\} \tag{13}$$

■

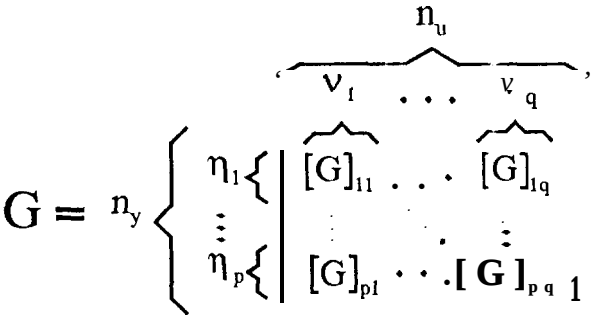

Figure 1: Construction of Partition $\mathcal{X} = \{p, q; \eta_1, \ldots, \eta_p; \nu_1, \ldots, \nu_q\}$.

EXAMPLE 1 Consider the matrix G ∈ 'R^{3X4},

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \end{bmatrix} \tag{14}$$

Define the partition $\mathcal{X} = \{2, 2; 2, 1; 2, 2\}$. Then the partitioned blocks $[G]_{ij} \in$ K(X, G' and corresponding index sets are given below,

$$[G]_{11} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}; \quad \kappa_{11}(\mathcal{X}) = \{(1,1),(1,2),(2,1),(2,2)\}$$

$$[G]_{12} = \begin{bmatrix} G_{13} & G_{14} \\ G_{23} & G_{24} \end{bmatrix}; \quad \kappa_{12}(\mathcal{X}) = \{(1,3),(1,4),(2,3),(2,4)\}$$

$$[G]_{21} = \begin{bmatrix} G_{31} & G_{32} \end{bmatrix}; \quad \kappa_{21}(\mathcal{X}) = \{(3,1), \quad \textbf{30)}\}$$

$$[G]_{22} = \{G_{33} \quad G34\ ]\ ,\ \kappa_{22}(\mathcal{X}) = \{(3,3),(3,4)\}$$

Partitioning is useful because it can be used to structure the additive uncertainty. Given a partition $\mathcal{X}$, one can define the plant blocks $[\mathcal{P}]_{ij} \in \mathcal{K}(\mathcal{X}, \mathcal{P})$ and nominal estimate blocks $[P^o]_{ij} \in \mathcal{K}(\mathcal{X}, P^o)$. Assume that a stable real rational filter $w_{ij}(z^{-1})$ is known which overbounds the error in the ijth block, i.e.,

$$|w_{ij}(e^{-j\omega T})| \geq \overline{\sigma}\left([\mathcal{P}]_{ij}(e^{-j\omega T}) - [P^o]_{ij}(e^{-j\omega T})\right) \text{ for all } \omega \in [0,\ \pi/T] \qquad (15),$$

Then one can write the additive error n structured form as,

$$\Delta_A = \begin{bmatrix} \Delta_{11}w_{11} & \cdots & \Delta_{1q}w_{1q} \\ \vdots & \ddots & \\ \Delta_{p1}w_{p1} & \cdots & \Delta_{pq}w_{pq} \end{bmatrix} \qquad (16)$$

for some set of norm bounded complex perturbations

$$\Delta_{ij} \in \mathcal{C}^{\eta_i \times \nu_j};\ \overline{\sigma}(\Delta_{ij}(\omega)) < 1; \quad \text{for all } \omega \in [0,\ \pi/T] \qquad (17)$$

Equivalently, one has,

$$\Delta\Delta = W_{A2}\Delta W_{A1} \qquad (18)$$

where,

$$W_{A2} = \begin{bmatrix} w_{11} \cdot I_{\eta_1} & \cdots & w_{1q} \cdot I_{\eta_1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \vdots & & \vdots \\ & & & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & w_{p1} \cdot I_{\eta_p} & \cdots & w_{pq} \cdot I_{\eta_p} \end{bmatrix} \in \mathcal{R}^{n_y \times (q \cdot n_y)} \qquad (19)$$

$$W_{A1} = \begin{bmatrix} I_{n_u} \\ \vdots \\ I_{n_u} \end{bmatrix} \in \mathcal{R}^{(p \cdot n_u) \times n_u} \qquad (20)$$

$$\Delta = diag[\Delta_{11}, \ldots, \Delta_{1q}, \ldots\ldots, \Delta_{p1}, \ldots, \Delta_{pq}] \in \mathcal{R}^{(q \cdot n_y) \times (p \cdot n_u)} \qquad (21)$$

where the notation I_n is used to denote the identity matrix of size $n \times n$. The error Δ in (18)(17) has the structure (21) which can be used with modern structured singular value analysis and p-synthesis methods.

3.2 MIMO PLANT SET ESTIMATION

The goal of this section is to determine the uncertainty in structured form (16)(18) using experimental data. This is equivalent to a plant set estimation problem, where the plant set is required to have a specific structure. The following definition introduces the notion of structured plant sets.

DEFINITION 3 A *structured additive uncertainty set* is defined as the set of plants $P \in \Omega_A(P^o, \mathcal{X}, \mathcal{L}_A)$ which is consistent with a partitioning $\mathcal{X}$, a specified nominal P^o, and uncertainty bounds $\mathcal{L}_A = \{\ell_{ij}(\omega), i = 1, ..., p, j = 1, ..., q\}$, i.e.,

$$\Omega_A(P^o, \mathcal{X}, \mathcal{L}_A) =$$

$$\{P : \left(\bar{\sigma}\, P_{ij} - P^o_{ij}\right) \leq \ell_{ij}(\omega), \text{ for all } \omega \in [0,\ \pi/T], [P]_{ij} \in \mathcal{K}(\mathcal{X}, P), [P^o]_{ij} \in \mathcal{K}(\mathcal{X}, P^o)\} \quad (22)$$

■

Simply stated, a candidate plant $P(z^{-1})$ is in the plant set $\Omega_A(P^o, \mathcal{X}, \mathcal{L}_A)$ if and only if it can be represented as,

$$\begin{bmatrix} [P]_{11}(\omega) & \cdots & [P]_{1q}(\omega) \\ \vdots & \ddots & \vdots \\ [P]_{p1}(\omega) & \cdots & [P]_{pq}(\omega) \end{bmatrix} =$$

$$\begin{bmatrix} [P^o]_{11}(\omega) & \cdots & [P^o]_{1q}(\omega) \\ \vdots & \ddots & \vdots \\ [P^o]_{p1}(\omega) & \cdots & [P^o]_{pq}(\omega) \end{bmatrix} + \begin{bmatrix} \Delta_{11}(\omega)\ell_{11}(\omega) & \cdots & \Delta_{1q}(\omega)\ell_{1q}(\omega) \\ \vdots & \ddots & \\ \Delta_{p1}(\omega)\ell_{p1}(\omega) & \cdots & \Delta_{pq}(\omega)\ell_{pq}(\omega) \end{bmatrix} \quad (23)$$

for some set of norm bounded perturbations $\bar{\sigma}(\Delta_{ij}(\omega)) < 1$, and for all $\omega \in [0, \pi/T]$.

The motivation for this definition is: if the true plant is known to lie in the set $\Omega_A(P^o, \mathcal{X}, \mathcal{L}_A)$, and if each scalar function $\ell_{ij}(\omega)$ is overbounded by a rational function $w_{ij}(z^{-1})$ of specified order, i.e.,

$$|w_{ij}(e^{-j\omega T})| \geq \ell_{ij}(\omega) \text{ for all } \omega \in (0, \pi/T] \quad (24)$$

then the true plant can be written in the desired structured uncertainty form (16)(18). An algorithm for systematically finding w_{ij} which satisfy (24) can be found in the next section on parametric structured uncertainty. .

In Definition 3, the plant set is characterized completely in terms of the bounds $\ell_{ij}(\omega)$ contained in $\mathcal{L}_A$. This notion is extended to *statistical* bounds in the following definition,

DEFINITION 4 The quantity, $\mathcal{L}_A^{1-\alpha} = \{\ell_{ij}(\omega), i = 1, ..., p\ j = 1, ..., q\}$ is said to be a *structured bound on the additive uncertainty with statistical confidence* $(1 - \alpha) \times 100\%$ with structure $\mathcal{X}$ if,

$$Prob\{\mathcal{P} \in \Omega_A(P^o, \mathcal{X}, \mathcal{L}_A^{1-\alpha})\} \geq 1 - \alpha \quad (25)$$

■

The significance of this definition is that $\mathcal{L}_A^{1-\alpha}$ characterizes (to statistical confidence $(1 - \alpha) \times 100\%$) a set in which the true plant $\mathcal{P}$ belongs. A statistical plant set description is useful since it

can systematically capture variations due to noise in the data. There is little to be lost from such an approach, since if a robust controller is designed to provide some specified level of performance *for all plants lying in the additive uncertainty set* $\Omega_A(P^o, \mathcal{X}, \mathcal{L}_A^{1-\alpha})$, then with probability $1 - \alpha$ the controller will work as planned when *applied to the true system.*

A method for calculating the statistical overbound $\mathcal{L}_A$ from noisy experimental is given in the next result.

THEOREM 1

Given discrete-time LTI plant $\mathcal{P}(z^{-1}) \in \mathcal{D}(M, \rho)$, assume that noisy frequency domain data $\{\hat{P}(\omega_k)\}_{k=1}^N$ are available on a grid on the unit circle O <w, , ..., $\omega_N < \pi/T$, where T is the sampling period.

Assume that a plant partition $\mathcal{X}$ is chosen, and let the accuracy of the data $[\hat{P}]_{ij} \in$ K(X, $\hat{P}$) for the ijth block be characterized by the quantity ϵ_{ij} (k) such that the event $E_{ij}(k)$,

$$E_{ij}(k): \quad \bar{\sigma}\left([\mathcal{P}]_{ij}(\omega_k) - [\hat{P}]_{ij}(\omega_k)\right) \leq \epsilon_{ij}(k) \tag{26}$$

is satisfied with at least probability $1 - \kappa$ at each grid point ω_k, and for each block i = 1, . . . p, $j = 1, ..., q$. Here, the events $E_{ij}(k)$ may or may not be statistically independent for different values of i, j, k.

Let $S([\hat{P}]_{ij}, \omega)$ be a *linear spline interpolant* to the data, i.e.,

$$S([\hat{P}]_{ij}, \omega) = \begin{cases} [\hat{P}]_{ij}(\omega_1) & \text{for } \omega \in [0, \omega_1] \\ [\hat{P}]_{ij}(\omega_k) + \frac{(\omega - \omega_k)}{\omega_{k+1} - \omega_k}\left([\hat{P}]_{ij}(\omega_{k+1}) - [\hat{P}]_{ij}(\omega_k)\right) & \text{for } \omega \in (\omega_k, \omega_{k+1}] \\ [\hat{P}]_{ij}(\omega_N) & \text{for } \omega \in (\omega_N, \pi/T] \end{cases} \tag{27}$$

and let $P^o(z^{-1})$ be any stable parametric model fit to the data, partitioned compatibly as [F'"]$_{ij} \in \mathcal{K}(\mathcal{X}$, P"). Define,

$$\mathcal{L}_A^{1-\alpha} = \{\ell_{ij}(\omega), i = 1, ..., p, j = 1, ..., q\} \tag{28}$$

where,

$$\ell_{ij}(\omega) = \mathcal{N}_{ij}(\omega) + \mathcal{C}_{ij}(\omega) \text{ 'i } \mathcal{I}_{ij}(\omega) \tag{29}$$

Noise Error

$$\mathcal{N}_{ij}(\omega) = \begin{cases} \epsilon_{ij}(1) & \text{for } \omega \in [0, \omega_1] \\ \epsilon_{ij}(k+1) & \text{for } \omega \in (\omega_k, \omega_{k+1}] \\ \epsilon_{ij}(N) & \text{for } \omega \in (\omega_N, \pi/T] \end{cases} \tag{30}$$

Curve Fit Error

$$\mathcal{C}_{ij}(\omega) = \bar{\sigma}\left(S([\hat{P}_{ij}], \omega) - [P^o]_{ij}(\omega)\right) \tag{31}$$

Interpolation Error

$$\mathcal{I}_{ij}(\omega) = \begin{cases} \bar{\sigma}\left(S([\hat{P}]_{ij}, \omega) - [\hat{P}]_{ij}(\omega_1)\right) + \frac{(\omega_1 - \omega)TM\rho}{(1-\rho)^2} & \text{for } \omega \in [0, \omega_1] \\ \bar{\sigma}\left(S([\hat{P}]_{ij}, \omega) - [\hat{P}]_{ij}(\omega_{k+1})\right) + \frac{(\omega_{k+1} - \omega)TM\rho}{(1-\rho)^2} & \text{for } \omega \in (\omega_k, \omega_{k+1}] \\ \bar{\sigma}\left(S([\hat{P}]_{ij}, \omega) - [\hat{P}]_{ij}(\omega_N)\right) + \frac{(\omega - \omega_N)TM\rho}{(1-\rho)^2} & \text{for } \omega \in (\omega_N, \pi/T] \end{cases} \tag{32}$$

Then, $\mathcal{L}_A^{1-\alpha}$ is a structured bound on the additive uncertainty with statistical confidence, $(1-\alpha)$ x 100% where,

$$1-\alpha = \begin{cases} (1-\kappa)^{Npq} & \text{if events } E_{ij}(k) \text{ are jointly independent for all i, } j,k \\ (1-N\kappa)^{pq} & \text{if events } E_{ij}(k) \text{ are jointly independent for all } i,j \\ (1-q\kappa)^{Np} & \text{if events } E_{ij}(k) \text{ are jointly independent for all } i,k \\ (1-p\kappa)^{Nq} & \text{if events } E_{ij}(k) \text{ are jointly independent for all } j,k \\ (1-qN\kappa)^{p} & \text{if events } E_{ij}(k) \text{ are jointly independent fOr all } i \\ (1-pN\kappa)^{q} & \text{if events } E_{ij}(k) \text{ are jointly independent for all } j \\ (1-pq\kappa)^{N} & \text{if events } E_{ij}(k) \text{ are jointly independent for all } k \\ (1-pqN\kappa) & \text{otherwise} \end{cases} \tag{33}$$

i.e., with the choice of $\mathcal{L}_A^{1-\alpha}$ in (28)(29) it follows that,

$$Prob\{\mathcal{P} \in \Omega_A(P^o, \mathcal{X}, \mathcal{L}_A^{1-\alpha})\} \geq 1-\alpha \tag{34}$$

PROOF: The proof can be found in Bayard [5].

REMARK 1: The various options for computing 1-- α in (33) of Theorem 1 can be very useful in practice. For example,

RI If data is taken in separate SIMO experiments, the noise entering each experiment will be statistically independent. Hence, the errors in each column of the estimated transfer function matrix will be statistically independent, and for any partition of the plant one can use the relation $(1-\alpha) = (1-pN\kappa)^q$. Furthermore, by suitable choice of partition, this approach can be extended to the case where data is taken in separate Ml MO experiments, each using a different *disjoint subset* of the available actuators.

R2 If the noise is known beforehand to be statistically independent from one sensor to the next (due to spatial remoteness, geometric considerations, orthogonal mountings, etc.), the errors in the rows of the estimated transfer function matrix will be statistically independent. Hence, for any partition of the plant one can use the relation $(1-\alpha) = (1-qN\kappa)^p$. By judicious choice of partition, this approach can be extended to the *case* where the noise is known to be independent between *disjoint subsets* of sensors.

R3 It is often the case (either exactly, or asymptotically), that errors incurred using frequency domain estimation techniques are statistically independent from one frequency to the next. In this case, one can use the relation $(1-\alpha) = (1-pq\kappa)^N$.

R4 If the above 3 situations (i.e., RI ,R2,R3) are valid simultaneously, one can use the relation $(1-\alpha) = (1-\kappa)^{Npq}$. Alternatively, the above cases can be combined 2-at-a-time to make use of the remaining relations (33), i.e., cases (R1 ,R2) imply that $(1-\alpha) = (1-N\kappa)^{pq}$; cases (R2,R3) imply that $(1-\alpha) = (1-q\kappa)^{Np}$; and cases (R1,R3) imply that $(1-\alpha) = (1-p\kappa)^{Nq}$.

R5 As a separate issue from statistical independence, it is often useful to define partitions which separate the transfer function into blocks having similar error magnitudes. With this approach, if the errors in a particular channel or subset of channels is much larger than the other channels, the remaining channels will not be unduly penalized in the robustness analysis. A similar argument can be made if the errors in any particular channel or subset of channels are much *smaller* than the other channels. Partitioning based on error magnitudes can be done with or without consideration of statistical independence between channels, by using the more general expressions for $1-\alpha$ given in (33).

As desired, $\mathcal{L}_A^{1-\alpha}$ in Theorem 1 is a function only of the experimental plant data set $\chi = \{M, \rho, \{\hat{P}(\omega_k), \epsilon_{ij}(k)\}_{ijk}\}$. Values for M and ρ will be assumed known a-priori (they may be known from the physics of the process, or found by impulse or step response experiments). Systematic methods for finding $\{\hat{P}(\omega_k), \epsilon_{ij}(k)\}_{ijk}$ with the desired properties in Theorem 1 under Assumptions 1-5, are given in Bayard [5].

4 PARAMETRIC STRUCTURED UNCERTAINTY

4.1 UNCERTAINTY CHARACTERIZATION

'I'his section presents an algorithm which solves the system posed by (24). That is, given the scalar function $\ell_{ij}(\omega)$, one wishes to determine a rational function $w_{ij}(z^{-1})$ of specified order to satisfy the inequality

$$|w_{ij}(e^{-j\omega T})| \geq \ell_{ij}(\omega) \quad \text{for all } \omega \in (0, \pi/T] \tag{35}$$

To accomplish this goal, a nonlinear constrained optimization is posed to compute a minimum-phase transfer function $w_{ij}(z^{-1})$ of order m such that $|w_{ij}(e^{-j\omega T})|$ is a tight overbound on $\ell_{ij}(\omega)$ for all ω. Forming the quantity $w_{ij}(z)w_{ij}(z^{-1})$ and evaluating on the unit circle gives an expression of the form,

$$w_{ij}(e^{j\omega T})w_{ij}(e^{-j\omega T}) = \frac{\beta^{ij}(\omega)}{\alpha(\omega)} \tag{36}$$

where,

$$\beta^{ij}(\omega) = \beta_0^{ij} + \beta_1^{ij}\cos(\omega T) + \ldots + \beta_m^{ij}\cos(m\omega T) \tag{37}$$

$$\alpha(\omega) = 1 + \alpha_1\cos(\omega T) + \ldots + \alpha_m\cos(m\omega T) \tag{38}$$

It is noted that $\alpha(\omega)$ is defined as monic without loss of generality (i.e., $\alpha_0 = 1$) . Here the indices i, j range over values determined by the partition defined by Definition 2. Thus, the complete parametric overbound is determined by the coefficients α_k, β_k^{ij} for $i = 1, \ldots, p, j = 1, \ldots, q$, $k = 1, \ldots, m$. Note that specifying $\alpha(\omega)$ to be a common denominator (i.e., independent of i and j) reduces the overall order of the parameterization.

Constraints for Overbounding

The requirement that $|w_{ij}(e^{-j\omega T})|$ be an overbound on $\ell_{ij}(\omega)$ is equivalent to the requirement that $|w_{ij}|^2$ is an overbound on ℓ_{ij}^2 and can be expressed as,

$$\frac{\beta^{ij}(\omega)}{\alpha(\omega)} \geq \ell_{ij}^2(\omega) \quad for\ all\ \omega \in [0, \pi/T] \tag{39}$$

Conditions for Tight Overbounding

The requirement that $|w_{ij}|^2$ be a "tight" overbound can be expressed as,

$$\min_{\alpha(\omega),\, \beta^{ij}(\omega)} \delta \tag{40}$$

$$i \in \{1, \ldots, p\}$$
$$j \in \{1, \ldots, q\}$$

where,

$$\delta = \max_{\omega}\{(\frac{\beta^{ij}(\omega)}{\alpha(\omega)} - \ell_{ij}^2(\omega))q_{ij}^{-1}(\omega)\} \tag{41}$$

Here, the criterion minimizes a worst-case error δ, which is frequency weighted by the non-negative quantity q_{ij}^{-1} (w). Note that the optimization here includes all channels simultaneously in one minimization problem $(i = 1, \ldots, p, j = 1, \ldots, q)$.

Constraints for Spectral **Factorizability**

The requirement that the overbound β^{ij}/α admits a spectral factor w_{ij} can be satisfied by ensuring that (Astrom [1])

$$\beta^{ij}(\omega)/\alpha(\omega) > 0 \quad for\ all\ w \in [0, \pi/T] \tag{42}$$

$$\text{cl(w)} > 0 \quad for\ all\ \omega \in [0, \pi/T] \tag{43}$$

Note that condition (42) is implied by (39), and condition (43) can be enforced explicitly by the constraint,

$$\alpha(\omega) \geq \underline{\alpha} > 0 \tag{44}$$

for some small $\underline{\alpha}$. For technical reasons, it will be convenient to enforce a similar constraint on β^{ij} as

$$\beta^{ij}(\omega) \geq \underline{\beta} > 0 \tag{45}$$

for some small $\underline{\beta}$.

In summary, it is desired to solve the optimization problem (40) (41) for $\alpha(\omega), \beta^{ij}(\omega)$ subject to constraints (39), (42) and (43).

4.2 Solution on a Finite Grid

The constrained optimization problem restricted to points of the set A can be written as,

$$\min_{\substack{\alpha_k, \beta_k^{ij} \\ i \in \{1, \ldots, p\} \\ j \in \{1, \ldots, q\} \\ k \in \{0, \ldots, m\}}} \delta \tag{46}$$

subject to

$$\beta^{ij}(\omega_\nu) - \ell_{ij}^2(\omega_\nu)\alpha(\omega_\nu) \geq 0 \tag{47}$$

$$\beta^{ij}(\omega_\nu) - \ell_{ij}^2(\omega_\nu)\alpha(\omega_\nu) \leq \delta q(\omega_\nu)\alpha(\omega_\nu) \tag{48}$$

$$\beta^{ij}(\omega_\nu) > \underline{\beta}; \quad \alpha(\omega_\nu) > \underline{\alpha} \tag{49}$$

$$for\ all\ i, j, \omega_\nu,\ (i = 1, \ldots, p; j = 1, \ldots, q; \nu = 1, \ldots, 72) \tag{50}$$

where $\alpha(\omega)$ and $\beta^{ij}(\omega)$ are defined by (37) (38). A key observation, made in [24], is that for fixed δ the optimization over α_k, β_k^{ij} is simply a linear programming problem to find a *feasible solution* for the coefficients $\alpha_k \beta_k^{ij}$. Hence, the joint optimization problem can be solved by a nested search

procedure where an outer-loop systematically decreases δ while an inner-loop finds feasible solutions in the variables α_k and β_k^{ij} for fixed δ. The procedure terminates when the smallest γ is found which admits a feasible solution. This approach is denoted as the **LP-Spectral** Overbounding and Factorization **(LPSOF)** Algorithm [24].

To solve the problem (12-13), one must begin with upper and lower bounds for the optimal value δ. For example, one can choose the lower bound $\delta_- = 0$ and let the upper bound δ_+ be derived from some starting feasible suboptimal solution (an obvious choice is $\alpha(\omega) \equiv 1$, $\beta^{ij}(\omega) \equiv \max_{i,j,\omega_\nu} \ell_{ij}^2(\omega_\nu)$). Then $\delta = (\delta_+ + \delta_-)/2$, becomes an updated value for δ_+ or δ_- depending on whether or not the inequalities (13) can be satisfied for $\delta = \hat{\delta}$ (i.e., the bisection method). In this way the LPSOF algorithm converges to the optimal value of δ geometrically, (i.e., as a power of 1 /2). Further refinements lead to additional speedup and other variations of the algorithm. More details of this approach can be found in Scheid [24], where the focus was on S1S0 systems.

5 STRUCTURED ROBUST CONTROL SYNTHESIS

Robust control methods such as H^2, $\mathbf{H}^\infty$ are well known design t echniques, applicable to systems having *unstructured* (norm-bounded, frequency dependent) uncertainty. In contrast, the μ-Synthsis design technique is best suited for systems having *structured* uncertainty representations. The μ-Synthesis approach achieves less conservative (i.e., higher performance) robust control designs by optimizing a robustness criteria based on a μ measure, or equivalently a structured (rather than unstructured) singular value definition [13].

This section describes the p-Synthesis technique [14], for the purpose of designing a controller which is robust with respect to the plant set, as identified in structured form using the estimation and overbounding techniques presented in earlier sections. Mathematically, the goal of p-Synthesis is to find a stabilizing controller $K(s)$ and a diagonal scaling matrix $D(s)$ such that

$$\| D T_{y_1 u_1} D^{-1} \|_\infty < 1$$

where $T_{y_1 u_1}$ is the matrix transfer function defined from the output of the additive uncertainty block Δ_A to its input, with the controller $K(s)$ in the loop (i.e., the transfer function "seen" by the uncertainty, with the controller active). The diagonal scaling matrix $D(s)$ is the key to closing the gap between the singular value (Small-Gain theorem) and the K_m function (Canonical Robust Control Problem) of the cost function. Readers are referred to [10] for detailed background.

An iterative design procedure called the 1)-K, Iteration will bring the solution of the robust control problem close to "optimal" (reducing the gap between singular value and μ). The procedure 'goes as follows:

Step 1: Set D(s) = I and use $\mathbf{H}^\infty$ control method to find a controller $K(s)$ which minimizes the cost function llDTY,U,D-ill@,

Step 2: Fix $K(s)$ and compute the *Structured Singular Value* and the cost-minimizing diagonal matrix D(s) (magnitude vs. frequency).

Step 3: Use a curve fitting method to realize a stable and minimal phase filter for the diagonal scaling D(s) over the frequency range of interest,

Step 4: If the structured singular value is close to and less than one, stop; otherwise go to Step 1,

6 EXAMPLE

A unified procedure for identification and control synthesis, has been developed in previous sections based on the 2-step approach of first estimating the plant set in a structured uncertainty form, and then utilizing a p-Synthesis routine to design a robust controller via the D-K iteration procedure. This approach will be applied to a numerical example in this section.

The example consists of two-cars connected with a spring and damper, as shown in Fig. 2.

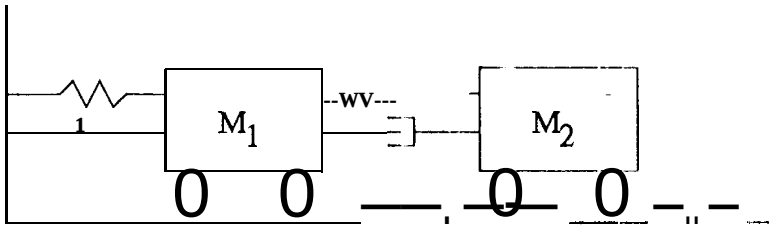

Figure 2: Two Car Problem.

A force actuator and position sensor are assumed to be collocated on each car, giving rise to a 2-input/2-output system. The maximum and minimum singular values of the open loop plant are shown in Fig. 3.

The objective is to compare a structured approach to plant set estimation using the fully partitioned plant $\chi_s = \{2,2; 1,1; 1,1\}$), with an unstructured approach using the unpartitioned plant $\chi_u = \{1, 1; 2; 2\}$. The control approach for the structured case will be p-synthesis, while the control approach for the unstructured case will be a pure H^∞ design.

Following the procedures of Sect. 3 and 4, the plant set is identified for the unpartitioned plant, and a stable and minimal phase filter W_2 is determined to overbound the unstructured additive error. A matrix weighting W_1 is chosen to specify a desired performance objective. Both weighings are appropriately used to augmented the open loop plant. With the given performance weighting W_1 and the additive weighting W_2, no controller could be found. using singular value mixed-sensitivity H^∞ synthesis.

Following the procedures of Sect. 3 and 4, the plant set is identified for the fully partitioned plant (i.e., $\chi_s = \{2,2; 1, 1; 1, 1\}$), and a set of additive uncertainties per each channel is realized by a set of stable and minimal phase filters (see Fig. 4). 'l'his set of filters (all with common denominator) are realized in matrix form to give a matrix weighting function W_2. The same performance weighting matrix W_1 is used as with the previous unparitioned case. The p-Synthesis procedure was applied to the problem, The results are summarized in Fig. 5. Not only is a robust controller found, but after two steps of the D-K iteration, the cost function shows an obvious improvement (the gap between μ and singular values is reduced).

7 CONCLUSIONS

This paper demonstrates a method for constructing a structured uncertainty model representation directly from experimental data, and then utilizing the structured model for robust control design. Specifically, this approach produces a nominal plant estimate P^o and the additive uncertainty weighting filters W_{A1}, W_{A2}, such that the true plant lies in the additive uncertainty set $\mathcal{P}$ = P“ +

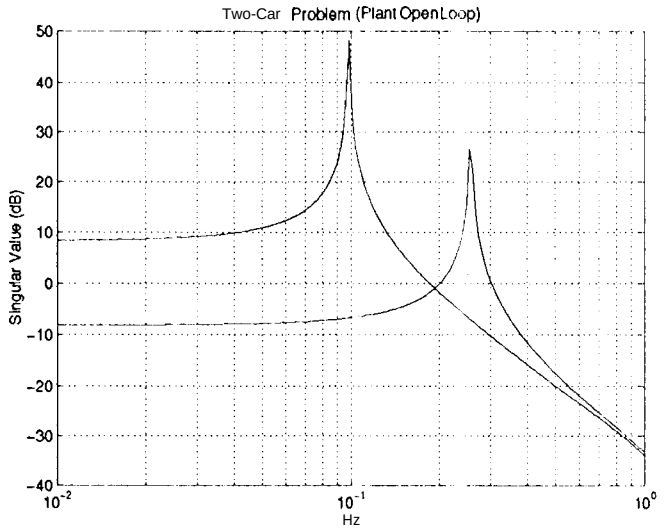

Figure 3: Two Car Problem (Plant Open Loop).

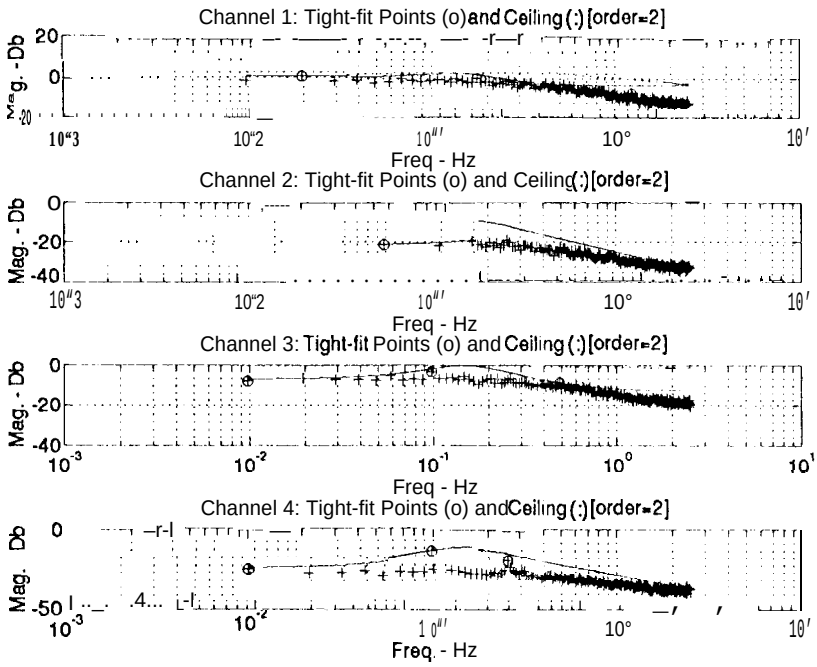

Figure 4: Additive Uncertainty and Overbound.

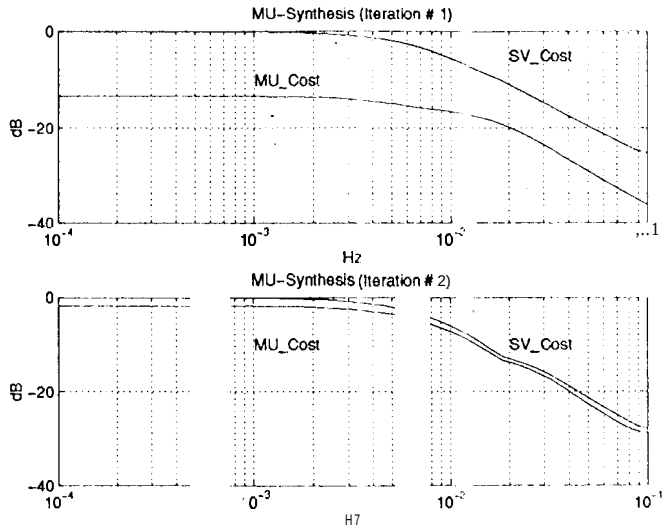

Figure 5: Singular Value and μ of the Cost Function.

$W_{A2} A W_{A1}$ to a prescribed statistical confidence $(1-\alpha)$ x 100%, where A can be structured as is appropriate to capture statistical independence and/or signal-to-noise variations between various channels of the transfer function matrix. The usefulness of this representation is that any controller designed to be robust with respect to P" and weights WA_1, W_{A2} (designed, for example with the software [10][2]), will work on the true system to the same $(1-\alpha)$ x 100% statistical confidence. The purpose of structured rather than unstructured bounds is to ensure tighter bounds on the estimation error and hence less conservative (i.e., higher performance) robust control designs.

8 ACKNOWLEDGEMENTS

This research was performed at the Jet propulsion Laboratory, California institute of Technology, under contract with the National Aeronautics and Space Administration.

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